Gemengde opgaven

9 Exponentiële en logaritmische functies

bladzijde 160

1 **a** $9^x = 3^x + 2$
$(3^x)^2 = 3^x + 2$
Stel $3^x = p$.
$p^2 = p + 2$
$p^2 - p - 2 = 0$
$(p-2)(p+1) = 0$
$p = 2 \vee p = -1$
$3^x = 2 \quad \vee \quad 3^x = -1$
$x = {}^3\log(2)$ geen opl.

b $\log^2(x) + 1 = 2\frac{1}{2}\log(x)$
Stel $\log(x) = p$.
$p^2 + 1 = 2\frac{1}{2}p$
$p^2 - 2\frac{1}{2}p + 1 = 0$
$(p-2)(p-\frac{1}{2}) = 0$
$p = 2 \vee p = \frac{1}{2}$
$\log(x) = 2 \vee \log(x) = \frac{1}{2}$
$x = 10^2 \vee x = 10^{\frac{1}{2}}$
$x = 100 \vee x = \sqrt{10}$

c $\frac{e^x}{e^x - 2} = 2$
$e^x = 2(e^x - 2)$
$e^x = 2e^x - 4$
$-e^x = -4$
$e^x = 4$
$x = \ln(4)$

d $\ln(3x+2) = \frac{1}{2}$
$3x + 2 = e^{\frac{1}{2}}$
$3x + 2 = \sqrt{e}$
$3x = -2 + \sqrt{e}$
$x = -\frac{2}{3} + \frac{1}{3}\sqrt{e}$

e $\ln(4x) - \ln(x+4) = 1$
$\ln\left(\frac{4x}{x+4}\right) = 1$
$\frac{4x}{x+4} = e$
$4x = ex + 4e$
$4x - ex = 4e$
$x(4 - e) = 4e$
$x = \frac{4e}{4-e}$

f $\ln^2(x-2)=4$
$\ln(x-2)=2 \vee \ln(x-2)=-2$
$x-2=\mathrm{e}^2 \vee x-2=\mathrm{e}^{-2}$
$x=2+\mathrm{e}^2 \vee x=2+\dfrac{1}{\mathrm{e}^2}$

g $3\cdot 2^{2x+1}+1=5\cdot 2^x$
$3\cdot 2^{2x}\cdot 2+1=5\cdot 2^x$
$6\cdot 2^{2x}+1=5\cdot 2^x$
$6\cdot (2^x)^2+1=5\cdot 2^x$
Stel $2^x=p$.
$6p^2+1=5p$
$6p^2-5p+1=0$
$D=25-4\cdot 6\cdot 1=1$, dus $\sqrt{D}=1$
$p=\dfrac{5-1}{12}=\frac{1}{3} \vee p=\dfrac{5+1}{12}=\frac{1}{2}$
$2^x=\frac{1}{2} \vee 2^x=\frac{1}{3}$
$x=-1 \vee x={}^2\log(\frac{1}{3})$

h $x\cdot 2^{-x+1}=4x\cdot 2^{-3x+1}$
$x=0 \vee 2^{-x+1}=4\cdot 2^{-3x+1}$
$x=0 \vee 2^{-x+1}=2^2\cdot 2^{-3x+1}$
$x=0 \vee 2^{-x+1}=2^{-3x+3}$
$x=0 \vee -x+1=-3x+3$
$x=0 \vee 2x=2$
$x=0 \vee x=1$

2 **a** $f(x)=x^2\mathrm{e}^{x-1}$ geeft $f'(x)=2x\cdot \mathrm{e}^{x-1}+x^2\cdot \mathrm{e}^{x-1}=(x^2+2x)\mathrm{e}^{x-1}$

b $y=\ln^2(x)=u^2$ met $u=\ln(x)$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}=2u\cdot\dfrac{1}{x}=2\ln(x)\cdot\dfrac{1}{x}=\dfrac{2\ln(x)}{x}$
$g(x)=\ln^2(x)+\ln(x^2)=\ln^2(x)+2\ln(x)$ geeft $g'(x)=\dfrac{2\ln(x)}{x}+2\cdot\dfrac{1}{x}=\dfrac{2\ln(x)+2}{x}$

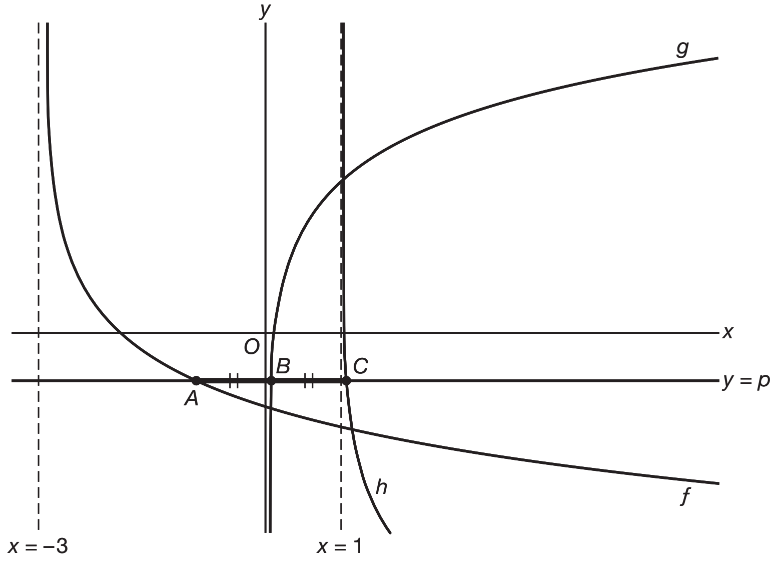
c $h(x)={}^2\log(x^3-x^2)={}^2\log(u)$ met $u=x^3-x^2$
$h'(x)=\dfrac{1}{u\ln(2)}\cdot(3x^2-2x)=\dfrac{3x^2-2x}{(x^3-x^2)\ln(2)}$

d $j(x)=\ln(\ln(2x))=\ln(u)$ met $u=\ln(2x)$
$j'(x)=\dfrac{1}{u}\cdot\dfrac{1}{x}=\dfrac{1}{\ln(2x)}\cdot\dfrac{1}{x}=\dfrac{1}{x\ln(2x)}$

3 **a** $y=2\,\mathrm{e}^x$
$\downarrow$ translatie (3, 0)
$y=2\,\mathrm{e}^{x-3}$
Er geldt $y=2\,\mathrm{e}^{x-3}=2\,\mathrm{e}^x\cdot \mathrm{e}^{-3}=\dfrac{1}{\mathrm{e}^3}\cdot 2\,\mathrm{e}^x$.
Dus de vermenigvuldiging ten opzichte van de x-as met $\dfrac{1}{\mathrm{e}^3}$ levert dezelfde beeldfiguur op.

b $y=\ln(2x)$
$\downarrow$ verm. y-as, 3
$y=\ln(2\cdot\frac{1}{3}x)=\ln(\frac{2}{3}x)$
Er geldt $y=\ln(\frac{2}{3}x)=\ln(\frac{1}{3}\cdot 2x)=\ln(\frac{1}{3})+\ln(2x)=\ln(2x)+\ln(\frac{1}{3})$.
De translatie $(0, \ln(\frac{1}{3}))$ levert dezelfde beeldfiguur op.

4 **a**



Stel de snijpunten met de grafieken van f, g en h zijn A, B en C.

$f(x) = p$ geeft ${}^{\frac{1}{3}}\log(x+3) = p$

$x + 3 = (\frac{1}{3})^p$

$x_A = -3 + (\frac{1}{3})^p$

$g(x) = p$ geeft $2 - {}^{\frac{1}{3}}\log(x) = p$

${}^{\frac{1}{3}}\log(x) = 2 - p$

$x_B = (\frac{1}{3})^{2-p}$

$h(x) = p$ geeft $-3 + {}^{\frac{1}{3}}\log(x-1) = p$

${}^{\frac{1}{3}}\log(x-1) = p+3$

$x - 1 = (\frac{1}{3})^{p+3}$

$x_C = 1 + (\frac{1}{3})^{p+3}$

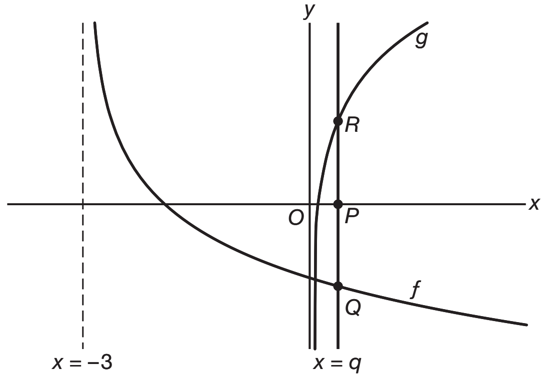
$AB = BC$ geeft $x_B = \frac{1}{2}(x_A + x_C)$

$(\frac{1}{3})^{2-p} = \frac{1}{2}(-3 + (\frac{1}{3})^p + 1 + (\frac{1}{3})^{p+3})$

$2 \cdot (\frac{1}{3})^{2-p} = -2 + (\frac{1}{3})^p + (\frac{1}{3})^{p+3}$

Voer in $y_1 = 2 \cdot (\frac{1}{3})^{2-x}$ en $y_2 = -2 + (\frac{1}{3})^x + (\frac{1}{3})^{x+3}$.
De optie intersect geeft $x \approx -0{,}43$.
Dus $p \approx -0{,}43$.

b



P is het midden van QR geeft $y_Q + y_R = 0$

$f(q) + g(q) = 0$

${}^{\frac{1}{3}}\log(q+3) + 2 - {}^{\frac{1}{3}}\log(q) = 0$

${}^{\frac{1}{3}}\log(q+3) + {}^{\frac{1}{3}}\log(\frac{1}{9}) = {}^{\frac{1}{3}}\log(q)$

${}^{\frac{1}{3}}\log((q+3) \cdot \frac{1}{9}) = {}^{\frac{1}{3}}\log(q)$

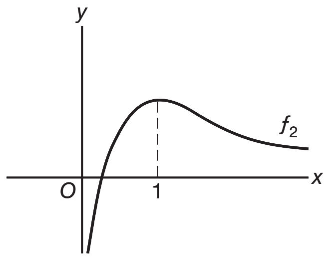
$\frac{1}{9}(q+3) = q$

$q+3=9q$

$-8q=-3$

$q=\frac{3}{8}$

5 **a** $f_2(x)=\dfrac{2+2\ln(x)}{x}$ geeft



$f_2'(x)=\dfrac{x\cdot 2\cdot\frac{1}{x}-(2+2\ln(x))\cdot 1}{x^2}=\dfrac{2-2-2\ln(x)}{x^2}=\dfrac{-2\ln(x)}{x^2}$

$f_2'(x)=0$ geeft $\dfrac{-2\ln(x)}{x^2}=0$

$-2\ln(x)=0$

$\ln(x)=0$

$x=1$

$f_2(1)=2$ geeft top (1, 2).

b $f_p(3)-f_{-p}(3)=4 \quad\vee\quad f_{-p}(3)-f_p(3)=4$

$\dfrac{2+p\ln(3)}{3}-\dfrac{2-p\ln(3)}{3}=4 \quad\vee\quad \dfrac{2-p\ln(3)}{3}-\dfrac{2+p\ln(3)}{3}=4$

$\dfrac{2p\ln(3)}{3}=4 \quad\vee\quad \dfrac{-2p\ln(3)}{3}=4$

$2p\ln(3)=12 \quad\vee\quad -2p\ln(3)=12$

$p=\dfrac{6}{\ln(3)} \quad\vee\quad p=-\dfrac{6}{\ln(3)}$

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6 **a** $f(x)=g(x)$ geeft $e^{\frac{1}{2}x-1}=e^{-x+1}$

$\frac{1}{2}x-1=-x+1$

$1\frac{1}{2}x=2$

$x=\frac{4}{3}$

$f(\frac{4}{3})=e^{\frac{2}{3}-1}=e^{-\frac{1}{3}}$, dus $A(\frac{4}{3}, e^{-\frac{1}{3}})$

$f(x)=e^{\frac{1}{2}x-1}$ geeft $f'(x)=\frac{1}{2}e^{\frac{1}{2}x-1}$

Stel $k: y=ax+b$ met $a=f'(\frac{4}{3})=\frac{1}{2}\cdot e^{-\frac{1}{3}}$.

$\left.\begin{array}{l} k: y=\frac{1}{2}e^{-\frac{1}{3}}x+b \\ \text{door } (\frac{4}{3}, e^{-\frac{1}{3}}) \end{array}\right\}\ e^{-\frac{1}{3}}=\frac{1}{2}\cdot e^{-\frac{1}{3}}\cdot\frac{4}{3}+b$

$\frac{1}{3}e^{-\frac{1}{3}}=b$

$k: y=\frac{1}{2}e^{-\frac{1}{3}}x+\frac{1}{3}e^{-\frac{1}{3}}$ snijden met de x-as geeft $\frac{1}{2}e^{-\frac{1}{3}}x+\frac{1}{3}e^{-\frac{1}{3}}=0$

$\frac{1}{2}x+\frac{1}{3}=0$

$x=-\frac{2}{3}$, dus $B(-\frac{2}{3}, 0)$

$g(x)=e^{-x+1}$ geeft $g'(x)=e^{-x+1}\cdot -1=-e^{-x+1}$

Stel $l: y=ax+b$ met $a=g'(\frac{4}{3})=-e^{-\frac{1}{3}}$.

$\left.\begin{array}{l} l: y=-e^{-\frac{1}{3}}x+b \\ \text{door } (\frac{4}{3}, e^{-\frac{1}{3}}) \end{array}\right\}\ e^{-\frac{1}{3}}=-e^{-\frac{1}{3}}\cdot\frac{4}{3}+b$

$2\frac{1}{3}e^{-\frac{1}{3}}=b$

$l: y=-e^{-\frac{1}{3}}x+2\frac{1}{3}e^{-\frac{1}{3}}$ snijden met de x-as geeft $-e^{-\frac{1}{3}}x+2\frac{1}{3}e^{-\frac{1}{3}}=0$

$-x+2\frac{1}{3}=0$

$x=2\frac{1}{3}$, dus $C(2\frac{1}{3}, 0)$

$O(\triangle ABC)=\frac{1}{2}\cdot(2\frac{1}{3}+\frac{2}{3})\cdot e^{-\frac{1}{3}}=\frac{3}{2}e^{-\frac{1}{3}}=\dfrac{3}{2\sqrt[3]{e}}$.

b $f(q) = g(q+3) = p \;\vee\; g(q) = f(q+3) = p$

$e^{\frac{1}{2}q-1} = e^{-q-3+1} \;\vee\; e^{-q+1} = e^{\frac{1}{2}(q+3)-1}$

$\frac{1}{2}q - 1 = -q - 2 \;\vee\; -q + 1 = \frac{1}{2}q + 1\frac{1}{2} - 1$

$1\frac{1}{2}q = -1 \;\vee\; -1\frac{1}{2}q = -\frac{1}{2}$

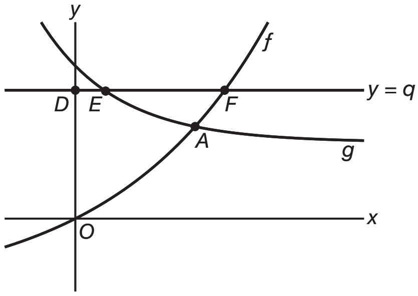
$q = -\frac{2}{3} \;\vee\; q = \frac{1}{3}$

$p = f(-\frac{2}{3}) = e^{-1\frac{1}{3}} \;\vee\; p = g(\frac{1}{3}) = e^{\frac{2}{3}}$

$PQ = 3$ voor $p = e^{-1\frac{1}{3}} = \dfrac{1}{e^{1\frac{1}{3}}} = \dfrac{1}{e \cdot \sqrt[3]{e}} \;\vee\; p = e^{\frac{2}{3}} = \sqrt[3]{e^2}$

$PQ < 3$ geeft $\dfrac{1}{e\sqrt[3]{e}} < p < \sqrt[3]{e^2}$

c



Stel $x_E = p$, dan is $x_F = 4p$.

$f(4p) = g(p) = q$ geeft $e^{\frac{1}{2}(4p)-1} = e^{-p+1}$

$2p - 1 = -p + 1$

$3p = 2$

$p = \frac{2}{3}$

$q = g(\frac{2}{3}) = e^{-\frac{2}{3}+1} = e^{\frac{1}{3}} = \sqrt[3]{e}$

7 **a** $f(x) = g(x)$ geeft $\ln(4x) = \ln\left(\dfrac{1}{x}\right)$

$4x = \dfrac{1}{x}$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = -\frac{1}{2} \;\vee\; x = \frac{1}{2}$

vold. niet voldoet

$f(\frac{1}{2}) = \ln(2)$ dus $A(\frac{1}{2}, \ln(2))$

$f(x) = \ln(4x)$ geeft $f'(x) = \dfrac{1}{x}$

$g(x) = \ln\left(\dfrac{1}{x}\right) = \ln(x^{-1}) = -\ln(x)$ geeft $g'(x) = -\dfrac{1}{x}$

Stel $y = ax + b$ met $a = f'(\frac{1}{2}) = 2$.

$y = 2x + b$
door $A(\frac{1}{2}, \ln(2))$ $\Big\}$ $\ln(2) = 2 \cdot \frac{1}{2} + b$
$\ln(2) - 1 = b$

$y = 2x + \ln(2) - 1$ snijdt de y-as in $(0, \ln(2) - 1)$.

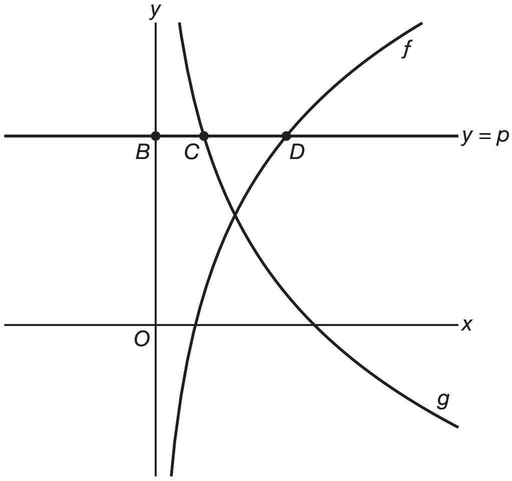
Stel $y = ax + b$ met $a = g'(\frac{1}{2}) = -2$.

$y = -2x + b$
door $A(\frac{1}{2}, \ln(2))$ $\Big\}$ $\ln(2) = -2 \cdot \frac{1}{2} + b$
$\ln(2) + 1 = b$

$y = 2x + \ln(2) + 1$ snijdt de y-as in $(0, \ln(2) + 1)$.

De lengte van het gevraagde lijnstuk is $\ln(2) + 1 - (\ln(2) - 1) = 2$.

b



Stel $x_C = q$, dan is $x_D = 2q$.

$f(2q) = g(q) = p$ geeft $\ln(8q) = \ln\left(\frac{1}{q}\right)$

$8q = \frac{1}{q}$

$8q^2 = 1$

$q^2 = \frac{1}{8}$

$q = \sqrt{\frac{1}{8}} \vee q = -\sqrt{\frac{1}{8}}$

voldoet   voldoet niet

$p = g(q) = \ln\left(\frac{1}{\sqrt{\frac{1}{8}}}\right) = \ln(\sqrt{8})$

8 **a** $f(x) = 2x$ geeft $x\mathrm{e}^{-x+1} = 2x$

$x\mathrm{e}^{-x+1} - 2x = 0$

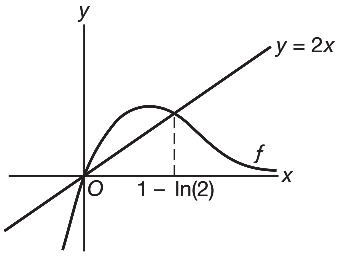
$x(\mathrm{e}^{-x+1} - 2) = 0$

$x = 0 \vee \mathrm{e}^{-x+1} = 2$

$x = 0 \vee -x + 1 = \ln(2)$

$x = 0 \vee -x = -1 + \ln(2)$

$x = 0 \vee x = 1 - \ln(2)$



$f(x) \leq 2x$ geeft $x \leq 0 \vee x \geq 1 - \ln(2)$

b $f(x) = x\mathrm{e}^{-x+1}$ geeft $f'(x) = 1 \cdot \mathrm{e}^{-x+1} + x \cdot -\mathrm{e}^{-x+1} = (1 - x)\mathrm{e}^{-x+1}$

$f'(0) = \mathrm{e}$, dus $y = \mathrm{e}x$ is raaklijn in O.

$x\mathrm{e}^{-x+1} = ax$ heeft precies één oplossing voor $a = \mathrm{e} \vee a \leq 0$.

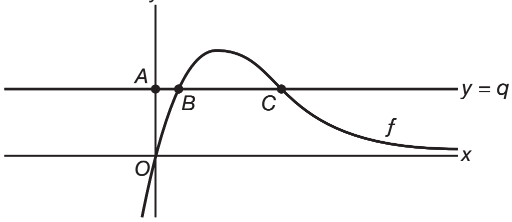
c $f(q) = f(q + 2)$ met $p = f(q)$ geeft $q\mathrm{e}^{-q+1} = (q + 2)\mathrm{e}^{-(q+2)+1}$ met $p = f(q)$

Voer in $y_1 = x\mathrm{e}^{-x+1}$ en $y_2 = (x + 2)\mathrm{e}^{-(x+2)+1}$.

De optie intersect geeft $x \approx 0{,}313$ en $y \approx 0{,}622$.

Dus $p \approx 0{,}62$.

d



Stel $x_B = p$, dan is $x_C = \mathrm{e} \cdot p$.

$f(p) = f(\mathrm{e} \cdot p) = q$ geeft $p\,\mathrm{e}^{-p+1} = \mathrm{e} \cdot p \cdot \mathrm{e}^{-\mathrm{e} \cdot p+1}$

$p = 0 \vee \mathrm{e}^{-p+1} = \mathrm{e} \cdot \mathrm{e}^{-\mathrm{e}p+1}$

vold. niet $\mathrm{e}^{-p+1} = \mathrm{e}^{-\mathrm{e}p+2}$

$-p + 1 = -\mathrm{e}p + 2$

$\mathrm{e}p - p = 1$

$p(\mathrm{e} - 1) = 1$

$p = \dfrac{1}{\mathrm{e}-1}$

$q = f(p) = f\left(\dfrac{1}{\mathrm{e}-1}\right) \approx 0{,}884$

9 **a** $f(x) = g_4(x)$ geeft $\mathrm{e}^x - 3 = 4\,\mathrm{e}^{-x}$

$\mathrm{e}^x - 3 = \dfrac{4}{\mathrm{e}^x}$

Stel $\mathrm{e}^x = p$.

$p - 3 = \dfrac{4}{p}$

$p^2 - 3p = 4$

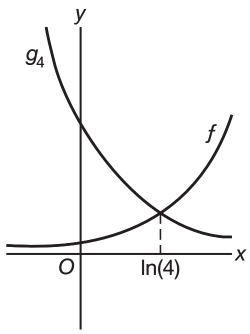
$p^2 - 3p - 4 = 0$

$(p + 1)(p - 4) = 0$

$p = -1 \vee p = 4$

$\mathrm{e}^x = -1 \vee \mathrm{e}^x = 4$

geen opl. $x = \ln(4)$



$f(x) < g_4(x)$ geeft $x < \ln(4)$

b $f(x) = \mathrm{e}^x - 3$ geeft $f'(x) = \mathrm{e}^x$

$f'(x) = 2$ geeft $\mathrm{e}^x = 2$

$x = \ln(2)$

$\left.\begin{array}{l} f(\ln(2)) = 2 - 3 = -1, \text{ dus raakpunt } (\ln(2), -1) \\ k\colon y = 2x + b \end{array}\right\}$ $-1 = 2\ln(2) + b$

$-1 - 2\ln(2) = b$

Dus $b = -1 - 2\ln(2)$.

c $g(x) = p\,\mathrm{e}^{-x}$ geeft $g'(x) = -p\,\mathrm{e}^{-x}$

$g'(x) = 2$ geeft $-p\,\mathrm{e}^{-x} = 2$

$p\,\mathrm{e}^{-x} = -2$

$\mathrm{e}^{-x} = -\dfrac{2}{p}$

$-x = \ln\left(-\dfrac{2}{p}\right)$

$x = -\ln\left(-\dfrac{2}{p}\right)$

$g_p\left(-\ln\left(-\frac{2}{p}\right)\right) = p \cdot -\frac{2}{p} = -2$, dus raakpunt $\left(-\ln\left(-\frac{2}{p}\right), -2\right)$
k: $y = 2x - 1 - 2\ln(2)$

$$
\begin{aligned}
-2 &= -2\ln\left(-\frac{2}{p}\right) - 1 - 2\ln(2) \\
2\ln\left(-\frac{2}{p}\right) &= 1 - 2\ln(2) \\
\ln\left(-\frac{2}{p}\right) &= \tfrac{1}{2} - \ln(2) \\
\ln\left(-\frac{2}{p}\right) &= \ln(e^{\frac{1}{2}}) - \ln(2) \\
\ln\left(-\frac{2}{p}\right) &= \ln(\sqrt{e}) - \ln(2) \\
\ln\left(-\frac{2}{p}\right) &= \ln\left(\frac{\sqrt{e}}{2}\right) \\
-\frac{2}{p} &= \frac{\sqrt{e}}{2} \\
p\sqrt{e} &= -4 \\
p &= -\frac{4}{\sqrt{e}}
\end{aligned}
$$

10 **a** $f(x) = 2$ geeft $\ln(x - 1) = 2$

$$
\begin{aligned}
x - 1 &= e^2 \\
x &= 1 + e^2
\end{aligned}
$$

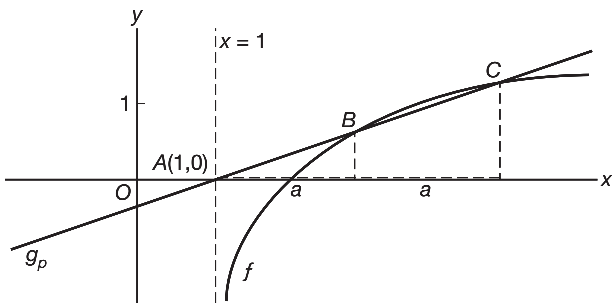
$g_p(1 + e^2) = 2$ geeft $p \cdot e^2 = 2$

$$p = \frac{2}{e^2}$$

b $g_p(x) = 0$ geeft $p(x - 1) = 0$

$$x = 1$$

Dus $A(1, 0)$.



B is het midden van AC geeft $f(1 + 2a) = 2 \cdot f(1 + a)$

$$
\begin{aligned}
&\ln(1 + 2a - 1) = 2 \cdot \ln(1 + a - 1) \\
&\ln(2a) = 2\ln(a) \\
&\ln(2a) = \ln(a^2) \\
&2a = a^2 \\
&a^2 - 2a = 0 \\
&a(a - 2) = 0 \\
&a = 0 \vee a = 2
\end{aligned}
$$

vold. niet voldoet

$a = 2$ geeft $x_B = 1 + a = 1 + 2 = 3$
$y_B = \ln(3 - 1) = \ln(2)$
$\Big\}$ $B(3, \ln(2))$

B op $y = p(x - 1)$ geeft $\ln(2) = p(3 - 1)$

$$
\begin{aligned}
\ln(2) &= 2p \\
p &= \tfrac{1}{2}\ln(2)
\end{aligned}
$$

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11 **a** $f_1(x) = \frac{\ln(x)}{x}$ geeft $f_1'(x) = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$

$f_1'(x) = 0$ geeft $1 - \ln(x) = 0$

$$
\begin{aligned}
\ln(x) &= 1 \\
x &= e
\end{aligned}
$$

$f(e) = \frac{1}{e}$, dus de top is $\left(e, \frac{1}{e}\right)$.

b $f_k(x) = \dfrac{\ln(kx)}{x}$ geeft $f_k'(x) = \dfrac{x \cdot \frac{1}{x} - \ln(kx) \cdot 1}{x^2} = \dfrac{1 - \ln(kx)}{x^2}$

$f_k'(x) = 0$ geeft $1 - \ln(kx) = 0$

$\ln(kx) = 1$

$\ln(kx) = 1$ geeft $y = \dfrac{1}{x}$

Dus voor $k \neq 0$ ligt de top van de grafiek van f_k op de kromme $y = \dfrac{1}{x}$.

c

k	y_1	y_2	x_A	x_B	AB
4	$\dfrac{\ln(4x)}{x}$	1	0,36	2,15	1,8
5	$\dfrac{\ln(5x)}{x}$	1	0,26	2,54	2,3

Dus vanaf $k = 5$ is $AB > 2$.

12 **a** 90% van 3,6 liter is 3,24 liter.

$L(t) = 3{,}24$ geeft $3{,}6(1 - e^{-2{,}5t}) = 3{,}24$

$1 - e^{-2{,}5t} = 0{,}9$

$-e^{-2{,}5t} = -0{,}1$

$e^{-2{,}5t} = 0{,}1$

$-2{,}5t = \ln(0{,}1)$

$t = \dfrac{\ln(0{,}1)}{-2{,}5} \approx 0{,}92$

Na ongeveer 0,9 seconde.