10 Integraalrekening

13 **a** $f(x) = 7 \cdot \log(5x) = 7 \cdot \log(5) + 7 \cdot \log(x)$ geeft
$F(x) = 7 \cdot \log(5) \cdot x + 7 \cdot \dfrac{x\ln(x) - x}{\ln(10)} + c$

b $f(x) = \mathrm{e}^{\frac{1}{2}x-1}$ geeft $F(x) = \dfrac{1}{\frac{1}{2}} \cdot \mathrm{e}^{\frac{1}{2}x-1} + c = 2\mathrm{e}^{\frac{1}{2}x-1} + c$

c $f(x) = 5\ln((2x-4)^2) = 10\ln(2x-4)$ geeft
$F(x) = \frac{10}{2}((2x-4)\ln(2x-4) - (2x-4)) + c = 5(2x-4)\ln(2x-4) - 5(2x-4) + c$

d $f(x) = \ln(x^2 - 6x + 9) = \ln((x-3)^2) = 2\ln(x-3)$ geeft
$F(x) = 2((x-3)\ln(x-3) - (x-3)) + c = 2(x-3)\ln(x-3) - 2(x-3) + c$

e $f(x)=\dfrac{x^4-6x^2\cdot\sqrt{x}+8x}{x^3}=\dfrac{x^4}{x^3}-\dfrac{6x^{2\frac{1}{2}}}{x^3}+\dfrac{8x}{x^3}=x-6x^{-\frac{1}{2}}+8x^{-2}$ geeft

$F(x)=\frac{1}{2}x^2-\dfrac{6}{\frac{1}{2}}x^{\frac{1}{2}}+\dfrac{8}{-1}\cdot x^{-1}+c=\frac{1}{2}x^2-12\sqrt{x}-\dfrac{8}{x}+c$

f $f(x)=(x^2+3)^2=x^4+6x^2+9$ geeft

$F(x)=\frac{1}{5}x^5+\frac{6}{3}x^3+9x+c=\frac{1}{5}x^5+2x^3+9x+c$

14 **a** $f(x)=\sqrt{6x+3}=(6x+3)^{\frac{1}{2}}$ geeft

$F(x)=\frac{1}{6}\cdot\dfrac{1}{1\frac{1}{2}}\cdot(6x+3)^{1\frac{1}{2}}+c=\frac{1}{9}(6x+3)^{1\frac{1}{2}}+c=\frac{1}{9}(6x+3)\sqrt{6x+3}+c$

b $f(x)=\dfrac{10}{2x-1}$ geeft $F(x)=\frac{1}{2}\cdot10\cdot\ln|2x-1|+c=5\ln|2x-1|+c$

c $f(x)=(3x-6)^{-2}$ geeft $F(x)=\frac{1}{3}\cdot\dfrac{1}{-1}\cdot(3x-6)^{-1}+c=-\frac{1}{3}\cdot(3x-6)^{-1}+c=\dfrac{-1}{3(3x-6)}+c$

d $f(x)=(2x+5)^{-1}=\dfrac{1}{2x+5}$ geeft $F(x)=\frac{1}{2}\ln|2x+5|+c$

e $f(x)=10^{2x-3}$ geeft $F(x)=\frac{1}{2}\cdot\dfrac{10^{2x-3}}{\ln(10)}+c=\dfrac{10^{2x-3}}{2\ln(10)}+c$

f $f(x)=\dfrac{8^x-1}{2^x}=\dfrac{8^x}{2^x}-\dfrac{1}{2^x}=4^x-2^{-x}$ geeft

$F(x)=\dfrac{4^x}{\ln(4)}-\dfrac{1}{-1}\cdot\dfrac{2^{-x}}{\ln(2)}+c=\dfrac{4^x}{\ln(4)}+\dfrac{2^{-x}}{\ln(2)}+c$

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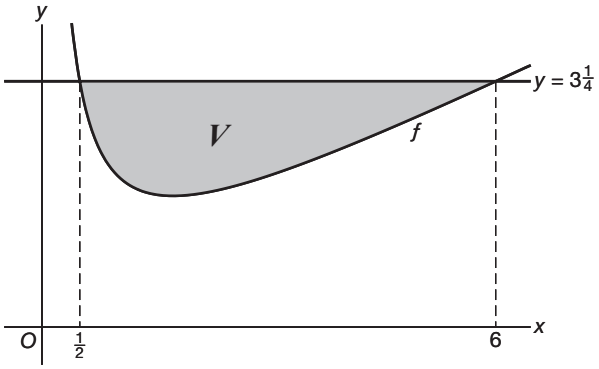
15 **a** $f(x)=3\frac{1}{4}$ geeft $\dfrac{x^2+3}{2x}=3\frac{1}{4}$

$x^2+3=6\frac{1}{2}x$

$x^2-6\frac{1}{2}x+3=0$

$(x-6)(x-\frac{1}{2})=0$

$x=6\ \vee\ x=\frac{1}{2}$

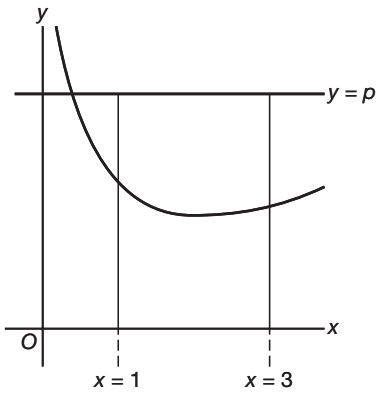


$O(V)=\displaystyle\int_{\frac{1}{2}}^{6}\left(3\tfrac{1}{4}-\frac{x^2+3}{2x}\right)\mathrm{d}x=\int_{\frac{1}{2}}^{6}\left(3\tfrac{1}{4}-\tfrac{1}{2}x-\frac{1\frac{1}{2}}{x}\right)\mathrm{d}x$

$=[3\frac{1}{4}x-\frac{1}{4}x^2-1\frac{1}{2}\ln(x)]_{\frac{1}{2}}^{6}=(19\frac{1}{2}-9-1\frac{1}{2}\ln(6))-(1\frac{5}{8}-\frac{1}{16}-1\frac{1}{2}\ln(\frac{1}{2}))$

$=8\frac{15}{16}-1\frac{1}{2}\ln(6)+1\frac{1}{2}\ln(\frac{1}{2})=8\frac{15}{16}-1\frac{1}{2}\ln(12)$

b



$\displaystyle\int_1^3 f(x)\,\mathrm{d}x=\int_1^3\frac{x^2+3}{2x}\,\mathrm{d}x=\int_1^3\left(\tfrac{1}{2}x+\frac{1\frac{1}{2}}{x}\right)\mathrm{d}x=[\tfrac{1}{4}x^2+1\tfrac{1}{2}\ln(x)]_1^3=(2\tfrac{1}{4}+1\tfrac{1}{2}\ln(3))-(\tfrac{1}{4}+1\tfrac{1}{2}\ln(1))=2+1\tfrac{1}{2}\ln(3)$

$\displaystyle\int_1^3 f(x)\,\mathrm{d}x=\tfrac{1}{2}\cdot O(\text{rechthoek})$ geeft $2+1\frac{1}{2}\ln(3)=\frac{1}{2}\cdot2p$

$p=2+1\frac{1}{2}\ln(3)$

16 **a** $f_p(x) = \frac{1}{3}x^3 + x^2 - 3x + p$ geeft $f_p'(x) = x^2 + 2x - 3$

Stel de raaklijn is $y = ax + b$.

$a = f_p'(0) = -3$, dus $y = -3x + b$
$f_p(0) = p$ dus door $(0, p)$
$\Big\}$ $p = 0 + b$, $p = b$

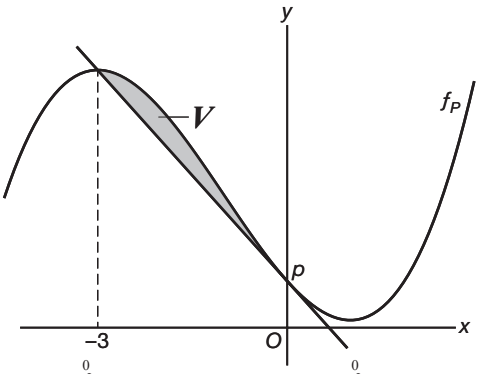
De raaklijn is $y = -3x + p$.

$f_p(x) = -3x + p$ geeft $\frac{1}{3}x^3 + x^2 - 3x + p = -3x + p$

$\frac{1}{3}x^3 + x^2 = 0$

$x^2(\frac{1}{3}x + 1) = 0$

$x = 0 \ \vee \ x = -3$



$O(V) = \int_{-3}^{0} (f_p(x) - (-3x + p))\,dx = \int_{-3}^{0} (\frac{1}{3}x^3 + x^2 - 3x + p + 3x - p)\,dx$

$= \int_{-3}^{0} (\frac{1}{3}x^3 + x^2)\,dx = [\frac{1}{12}x^4 + \frac{1}{3}x^3]_{-3}^{0} = 0 - (\frac{81}{12} - 9) = 2\frac{1}{4}$

b $f_p'(x) = 0$ geeft $x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3 \ \vee \ x = 1$

$x_A > 0$, dus $x_A = 1$

$f_p(1) = 0$ geeft $\frac{1}{3} \cdot 1^3 + 1^2 - 3 \cdot 1 + p = 0$

$p = 1\frac{2}{3}$

Voer in $y_1 = \frac{1}{3}x^3 + x^2 - 3x + 1\frac{2}{3}$.

De optie zero (TI) of ROOT (Casio) geeft $x = -5$ en $x = 1$.

De optie fnInt (TI) of $\int dx$ (Casio) geeft $O(V) = \int_{-5}^{1} f_{1\frac{2}{3}}(x)\,dx \approx 36{,}000$.

17 **a** $I = \int_{\frac{1}{3}r}^{\frac{1}{2}r} \pi y^2\,dx = \int_{\frac{1}{3}r}^{\frac{1}{2}r} \pi(r^2 - x^2)\,dx = [\pi(r^2x - \frac{1}{3}x^3)]_{\frac{1}{3}r}^{\frac{1}{2}r} = \pi(r^2 \cdot \frac{1}{2}r - \frac{1}{3}(\frac{1}{2}r)^3) - \pi(r^2 \cdot \frac{1}{3}r - \frac{1}{3}(\frac{1}{3}r)^3)$

$= \pi(\frac{1}{2}r^3 - \frac{1}{24}r^3) - \pi(\frac{1}{3}r^3 - \frac{1}{81}r^3) = \frac{11}{24}\pi r^3 - \frac{26}{81}\pi^3 = \frac{89}{648}\pi r^3$

b $I(\text{bol}) = \frac{4}{3}\pi r^3$

$I = \int_{-pr}^{pr} \pi y^2\,dx = \int_{-pr}^{pr} \pi(r^2 - x^2)\,dx = [\pi(r^2x - \frac{1}{3}x^3)]_{-pr}^{pr} = \pi(r^2 \cdot pr - \frac{1}{3}(pr)^3) - \pi(r^2 \cdot (-pr) - \frac{1}{3}(-pr)^3)$

$= p\pi r^3 - \frac{1}{3}p^3\pi r^3 + p\pi r^3 - \frac{1}{3}p^3\pi r^3 = (2p - \frac{2}{3}p^3)\pi r^3$

$I = \frac{1}{2}I(\text{bol})$ geeft $(2p - \frac{2}{3}p^3)\pi r^3 = \frac{2}{3}\pi r^3$

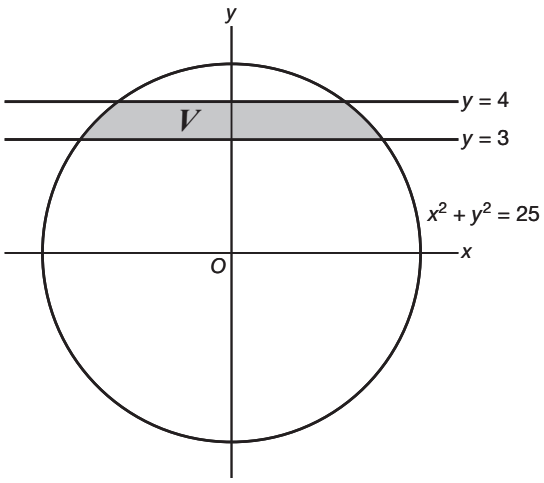
$2p - \frac{2}{3}p^3 = \frac{2}{3}$

Voer in $y_1 = 2x - \frac{2}{3}x^3$ en $y_2 = \frac{2}{3}$.

De optie intersect geeft $x \approx 0{,}35$

Dus $p \approx 0{,}35$.

18



Substitutie van $y = 3$ in $x^2 + y^2 = 25$ geeft $x^2 + 9 = 25$

$x^2 = 16$

$x^2 = 4 \ \vee \ x = -4$

Substitutie van $y = 4$ in $x^2 + y^2 = 25$ geeft $x^2 + 16 = 25$

$x^2 = 9$

$x = 3 \ \vee \ x = -3$

$$I = \int_{-4}^{-3} \pi y^2 \, dx + \int_{-3}^{3} \pi \cdot 4 \, dx + \int_{3}^{4} \pi \cdot y^2 \, dx - \int_{-4}^{4} \pi \cdot 3 \, dx$$

$$= \int_{-4}^{-3} \pi(25 - x^2) \, dx + [4\pi x]_{-3}^{3} + \int_{3}^{4} \pi(25 - x^2) \, dx - [3\pi x]_{-4}^{4}$$

$$= [\pi(25x - \tfrac{1}{3}x^3)]_{-4}^{-3} + (12\pi - -12\pi) + [\pi(25x - \tfrac{1}{3}x^3)]_{3}^{4} - (12\pi - -12\pi)$$

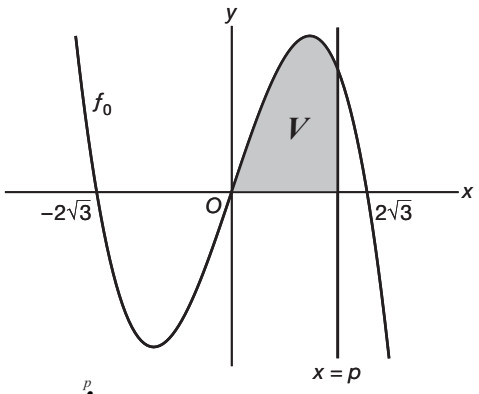
$$= \pi(-75 + 9) - \pi(-100 + \tfrac{64}{3}) + 24\pi + \pi(100 - \tfrac{64}{3}) - \pi(75 - 9) - 24\pi$$

$$= -66\pi + 78\tfrac{2}{3}\pi + 78\tfrac{2}{3}\pi - 66\pi = 25\tfrac{1}{3}\pi$$

19 **a** $f_0(x) = 0$ geeft $-\frac{1}{3}x^3 + 4x = 0$

$-\frac{1}{3}x(x^2 - 12) = 0$

$x = 0 \ \vee \ x = \sqrt{12} = 2\sqrt{3} \ \vee \ x = -2\sqrt{3}$



$$O(V) = \int_{0}^{p} (-\tfrac{1}{3}x^3 + 4x) \, dx = [-\tfrac{1}{12}x^4 + 2x^2]_0^p = (-\tfrac{1}{12}p^4 + 2p^2) - 0 = -\tfrac{1}{12}p^4 + 2p^2$$

$O(V) = 10$ geeft $-\frac{1}{12}p^4 = 2p^2 = 10$

$-\frac{1}{12}p^4 + 2p^2 - 10 = 0$

$p^4 - 24p^2 + 120 = 0$

Stel $q = p^2$.

$q^2 - 24q + 120 = 0$

$D = (-24)^2 - 4 \cdot 1 \cdot 120 = 96$, dus $\sqrt{D} = \sqrt{96} = 4\sqrt{6}$

$$q = \frac{24 - 4\sqrt{6}}{2} = 12 - 2\sqrt{6} \ \vee \ q = 12 + 2\sqrt{6}$$

$p^2 = 12 - 2\sqrt{6} \ \vee \ p^2 = 12 + 2\sqrt{6}$

$p = \sqrt{12 - 2\sqrt{6}} \ \vee \ p = -\sqrt{12 - 2\sqrt{6}} \ \vee \ p = \sqrt{12 + 2\sqrt{6}} \ \vee \ p = -\sqrt{12 + 2\sqrt{6}}$

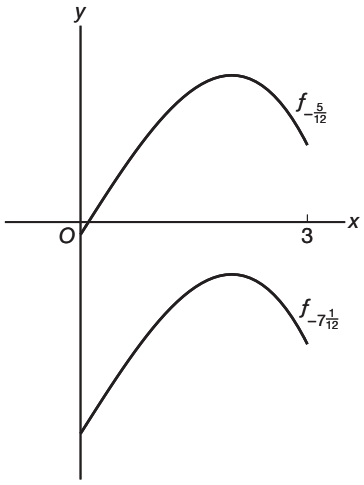
$0 < p < 2\sqrt{3}$, dus $p = \sqrt{12 - 2\sqrt{6}}$.

b $\int_0^3 (-\frac{1}{3}x^3 + 4x + a)\,dx = [-\frac{1}{12}x^4 + 2x^2 + ax]_0^3 = (-6\frac{3}{4} + 18 + 3a) - 0 = 3a + 11\frac{1}{4}$

$O(W) = 10$ geeft $3a + 11\frac{1}{4} = 10 \vee 3a + 11\frac{1}{4} = -10$

$3a = -1\frac{1}{4} \vee 3a = -21\frac{1}{4}$

$a = -\frac{5}{12} \vee a = -7\frac{1}{12}$



Dus $a = -7\frac{1}{12}$

bladzijde 165

20 **a** $f(x) = \frac{1}{4}x^2$ geeft $f'(x) = \frac{1}{2}x$

$f'(x) = 1$ geeft $\frac{1}{2}x = 1$

$x = 2$, dus raakpunt $A(2, 1)$

$\left.\begin{array}{l} y = x + b \\ A(2, 1) \end{array}\right\}$ $1 = 2 + b$

$-1 = b$

Dus $y = x - 1$.

$g(x) = -\frac{4}{x^2} = -4x^{-2}$ geeft $g'(x) = 8x^{-3} = \frac{8}{x^3}$

$g'(x) = 1$ geeft $\frac{8}{x^3} = 1$

$x^3 = 8$

$x = 2$, dus raakpunt $B(2, -1)$

$\left.\begin{array}{l} y = x + b \\ B(2, -1) \end{array}\right\}$ $-1 = 2 + b$

$-3 = b$

Dus $y = x - 3$.

De raaklijnen snijden de y-as in $(0, -1)$ en $(0, -3)$. De diagonaal van het vierkant is dus 2.

b $f(a) = \frac{1}{4}a^2$ dus $C(a, \frac{1}{4}a^2)$ en $D(-a, \frac{1}{4}a^2)$

$g(a) = -\frac{4}{a^2}$ dus $B\left(a, -\frac{4}{a^2}\right)$ en $A\left(-a, -\frac{4}{a^2}\right)$

$\text{opp}(ABCD) = 2a \cdot \left(\frac{1}{4}a^2 + \frac{4}{a^2}\right) = \frac{1}{2}a^3 + \frac{8}{a}$

$\text{opp}(\text{vlakdeel boven de grafiek van } f) = 2 \cdot \int_0^a (\frac{1}{4}a^2 - \frac{1}{4}x^2)\,dx = 2 \cdot [\frac{1}{4}a^2x - \frac{1}{12}x^3]_0^a = 2(\frac{1}{4}a^3 - \frac{1}{12}a^3) = \frac{1}{3}a^3$

$\text{opp}(\text{vlakdeel boven de grafiek van } f) = \frac{1}{2} \cdot \text{opp}(ABCD)$ geeft $\frac{1}{3}a^3 = \frac{1}{2} \cdot \left(\frac{1}{2}a^3 + \frac{8}{a}\right)$

$\frac{1}{3}a^3 = \frac{1}{4}a^3 + \frac{4}{a}$

$\frac{1}{12}a^3 = \frac{4}{a}$

$a^4 = 48$, dus $a = \sqrt[4]{48}$

21 **a** $A(0, 1)$ en $B(1, e^{-1}) = (1, \frac{1}{e})$

$\text{rc}_{AB} = \frac{\frac{1}{e} - 1}{1} = \frac{1}{e} - 1$

$AB: y = \left(\frac{1}{e} - 1\right)x + b$ door (0, 1) $\Big\}$ $1 = 0 + b$, $1 = b$

Dus $AB: y = \left(\frac{1}{e} - 1\right)x + 1$.

$O(V) = \int_0^1 \left(\left(\frac{1}{e} - 1\right)x + 1\right) dx - \int_0^1 e^{-x}\, dx = \left[\frac{1}{2}\left(\frac{1}{e} - 1\right)x^2 + x\right]_0^1 - [-e^{-x}]_0^1 = \frac{1}{2}\left(\frac{1}{e} - 1\right) \cdot 1^2 + 1 - (-e^{-1} - -e^0)$

$= \frac{1}{2e} - \frac{1}{2} + 1 + e^{-1} - 1 = \frac{1}{2e} - \frac{1}{2} + \frac{1}{e} = \frac{1}{2e} - \frac{1}{2} + \frac{2}{2e} = \frac{3}{2e} - \frac{1}{2}$

b $AB = \sqrt{1^2 + \left(\frac{1}{e} - 1\right)^2} \approx 1{,}183$

$f(x) = e^{-x}$ geeft $f'(x) = -e^{-x}$

De optie fnInt (TI) of $\int dx$ (Casio) geeft boog$(AB) = \int_0^1 \sqrt{1 + (-e^{-x})^2}\, dx \approx 1{,}193$.

omtrek$(V) \approx 1{,}183 + 1{,}193 \approx 2{,}38$

c $f'(x) = \text{rc}_{AB}$ geeft $-e^{-x} = \frac{1}{e} - 1$

Voer in $y_1 = -e^{-x}$ en $y_2 = \frac{1}{e} - 1$.

De optie intersect geeft $x \approx 0{,}46$.

Dus $x_C \approx 0{,}46$.

bladzijde 166

22 **a** $O = \int_0^2 3(2x - x^2)\, dx - \int_0^2 2(2x - x^2)\, dx = [3(x^2 - \frac{1}{3}x^3)]_0^2 - [2(x^2 - \frac{1}{3}x^3)]_0^2$

$= 3(4 - \frac{8}{3}) - 2(4 - \frac{8}{3}) = 4 - 2\frac{2}{3} = 1\frac{1}{3}$

b $x = 1{,}99$ invullen in $n(2x - x^2) = x$ geeft $n \cdot (2 \cdot 1{,}99 - 1{,}99^2) = 1{,}99$

$n \cdot 0{,}0199 = 1{,}99$

$n = 100$

Dus voor $n > 100$ is $x_{S_n} > 1{,}99$.

c $y = n(2x - x^2) = 2nx - nx^2$ geeft $\frac{dy}{dx} = 2n - 2nx$

$\left[\frac{dy}{dx}\right]_{x=0} = 2n$

De raaklijn in O is dus $y = 2nx$.

$x = 1$ geeft $y = 2n \cdot 1 = 2n$, dus $R_n = (1, 2n)$.

$x = 1$ invullen in $y = n(2x - x^2)$ geeft $y = n \cdot (2 \cdot 1 - 1^2) = n \cdot 1 = n$, dus $T_n = (1, n)$

Dus $A(1, 0)$, $T_n(1, n)$ en $R_n(1, 2n)$.

T_n is het midden van AR_n omdat $\frac{y_A + y_{R_n}}{2} = \frac{0 + 2n}{2} = n = y_{T_n}$.

bladzijde 167

23 **a** $O = 1 \cdot e^{a+1} - \int_a^{a+1} e^x\, dx = e^{a+1} - [e^x]_a^{a+1} = e^{a+1} - (e^{a+1} - e^a) = e^{a+1} - e^{a+1} + e^a = e^a$

$O = 3$ geeft $e^a = 3$

$a = \ln(3)$

b $A(a, e^a)$ en $B(a + 1, e^{a+1})$

$\text{rc}_{AB} = \frac{e^{a+1} - e^a}{a + 1 - a} = e^{a+1} - e^a$

$\text{rc}_{AB} < 1$ geeft $e^{a+1} - e^a < 1$

Voer in $y_1 = e^{x+1} - e^x$ en $y_2 = 1$.

De optie intersect geeft $x \approx -0{,}54$.

Dus $a < -0{,}54$.

c $f(x) = e^x$ geeft $f'(x) = e^x$

De optie fnInt (TI) of $\int dx$ (Casio) geeft lengte $= \int_0^2 \sqrt{1 + (e^x)^2}\, dx \approx 4{,}79$.

d $OPAQ$ wentelen om de x-as geeft $I_1 = \int_0^1 \pi e^2\, dx = [\pi e^2 x]_0^1 = \pi e^2$.

Het deel onder de grafiek van f wentelen geeft

$I_2 = \int_0^1 \pi(e^x)^2\, dx = \int_0^1 \pi e^{2x}\, dx = [\frac{1}{2}\pi e^{2x}]_0^1 = \frac{1}{2}\pi e^2 - \frac{1}{2}\pi$.

De inhoud van het deel boven de grafiek van f wentelen is

$I_1 - I_2 = \pi e^2 - (\frac{1}{2}\pi e^2 - \frac{1}{2}\pi) = \frac{1}{2}\pi e^2 + \frac{1}{2}\pi$.

Het verschil tussen de inhouden is $\frac{1}{2}\pi e^2 + \frac{1}{2}\pi - (\frac{1}{2}\pi e^2 - \frac{1}{2}\pi) = \pi$.

bladzijde 168

24 **a** De optie fnInt (TI) of $\int dx$ (Casio) geeft $O(G) = \int_0^3 0{,}9x^{1\frac{1}{2}}\,dx - \int_0^3 0{,}5x^{1\frac{1}{2}}\,dx \approx 2{,}49$.

b $I = \int_0^3 \pi(2x^{1\frac{1}{2}})^2\,dx = \int_0^3 4\pi x^3\,dx = [\pi x^4]_0^3 = 81\pi$

c $f_{\frac{2}{3}}(x) = \frac{2}{3}x^{1\frac{1}{2}}$ geeft $f'_{\frac{2}{3}}(x) = \frac{2}{3}\cdot 1\frac{1}{2}\cdot x^{\frac{1}{2}} = \sqrt{x}$

$L\left(\frac{2}{3}\right) = \int_0^3 \sqrt{1+(\sqrt{x})^2}\,dx = \int_0^3 \sqrt{1+x}\,dx = \int_0^3 (1+x)^{\frac{1}{2}}\,dx = \left[\frac{1}{1\frac{1}{2}}(1+x)^{1\frac{1}{2}}\right]_0^3 = [\frac{2}{3}(x+1)\sqrt{x+1}]_0^3$

$= \frac{2}{3}\cdot 4\cdot\sqrt{4} - \frac{2}{3}\cdot 1\cdot\sqrt{1} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$